

Vacuum polarization in $d + \frac{1}{2}$ dimensions

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Abstract

We study the main properties of the one-loop vacuum polarization function ($\Pi_{\alpha\beta}$) for spinor QED in ' $d + \frac{1}{2}$ dimensions', i.e., with fields defined on $\mathcal{M} \subset \mathbb{R}^{d+1}$ such that $\mathcal{M} = \{(x_0, \dots, x_d) | x_d \geq 0\}$, with bag-like boundary conditions on the boundary $\partial\mathcal{M} = \{(x_0, \dots, x_d) | x_d = 0\}$. We obtain an exact expression for the induced current due to an external constant electric field normal to the boundary. We show that, for the particular case of $2 + 1$ dimensions, there is a transverse component for the induced current, which is localized on a region close to $\partial\mathcal{M}$. This current is a parity breaking effect purely due to the boundary.

There are many interesting quantum field theory models where the presence of boundaries is relevant to the physical properties of the system. Noteworthy examples are the Casimir effect [1, 2], the bag model of QCD [3], as well as many condensed matter physics models [4].

One of the most interesting consequences of the presence of boundaries in a quantum field theory is that they strongly influence the structure of the quantum fluctuations. Besides, response functions are determined by the correlation between fluctuations; therefore, those functions should also be quite sensitive to the presence of boundaries. In particular, one should

expect a strong effect on the QED vacuum polarization function, since a bag-like condition implies that the normal component of the current due to an external field vanishes on the boundary. Since that current can be expressed in terms of the vacuum polarization function, the latter should depend strongly on the distance between its spatial arguments and $\partial\mathcal{M}$. In other words, the conductivity tensor becomes space-dependent and also anisotropic in the presence of a bag-like condition.

In this letter, we calculate the vacuum polarization function for massless fermions in a $d+1$ -dimensional region \mathcal{M} , defined by the conditions $x_d \geq 0$ ¹. Equipped with the general expression for $\Pi_{\alpha\beta}$, we then derive the induced current when an external constant electric field is applied in the direction normal to the ‘wall’ at $x_d = 0$. We shall show that, for the case $d = 2$, there is an induced current in the x_1 direction when a constant electric field in the x_2 direction (i.e., normal to the wall) is applied. This is a parity-breaking effect due to the fact that the spatial region itself is not invariant under parity transformations. We recall that, in $2+1$ dimensions, those transformations are really reflections about an axis [5].

The calculation of $\Pi_{\alpha\beta}$ in the standard perturbation theory approach (which we shall follow) requires the knowledge of the fermion propagator in a region with boundaries. To that end, the multiple reflection expansion (MRE) [6, 7, 8, 9] provides a systematic approach to the calculation of quantum field theory propagators in a spacetime region \mathcal{M} , with a non-trivial boundary $\mathcal{S} \equiv \partial\mathcal{M}$. The different terms in the MRE are constructed with the standard (no-boundary) free propagator, and can be conveniently ordered according to the increasing number of ‘interactions’ with \mathcal{S} .

S_F is determined by the following equations:

$$\begin{cases} (\not{\partial} + m)S_F(x, y) = \delta^{(d+1)}(x - y) & , \quad \forall x, y \in \mathcal{M} \\ \lim_{x \rightarrow \alpha} P_\alpha S_F(x, y) = 0 & , \quad \forall \alpha \in \mathcal{S}, \quad \forall y \in \mathcal{M} \end{cases} \quad (1)$$

where $P_\alpha \equiv \frac{1+\gamma \cdot n(\alpha)}{2}$, $n(\alpha)$ is the (outer) normal at a given point $\alpha \in \mathcal{S}$, and the $x \rightarrow \alpha$ limit should be taken *from inside* \mathcal{M} .

We shall assume that the fermions are massless. In this case, it can be shown that the Dirac propagator in $d + \frac{1}{2}$ dimensions, S_F , contains only one reflection. To write S_F explicitly, it is convenient to take into account the translation invariance with respect to the first d coordinates $x_\perp = (x_0, x_1, \dots, x_{d-1})$, by using a mixed Fourier transformation: denoting

¹Following [4] we use the expression ‘ $d + \frac{1}{2}$ dimensions’ to denote such a spacetime.

the x coordinates in \mathbb{R}^{d+1} by $x = (x_\perp; s)$, while $s \equiv x_d$, the ‘mixed’ Fourier transform \tilde{f} of a function $f(x_\perp; s)$ will be determined by,

$$f(x_\perp; s) = \int \frac{d^d p}{(2\pi)^d} e^{ip \cdot x_\perp} \tilde{f}(p; s), \quad (2)$$

where p will denote a momentum vector (we shall omit writing the label \perp for d -dimensional vectors like p wherever there is no risk of confusion).

Then the explicit solution for \tilde{S}_F is:

$$\tilde{S}_F(p; s, s') = \tilde{S}_F^{(0)}(p; s, s') + \tilde{S}_F^{(1)}(p; s, s'), \quad (3)$$

where,

$$\tilde{S}_F^{(0)}(p; s, s') = \frac{1}{2} [\gamma_d \text{sgn}(s - s') - i \not{p}] e^{-|p| |s - s'|} \quad (4)$$

and

$$\begin{aligned} \tilde{S}_F^{(1)}(p; s, s') &= -\tilde{S}_F^{(0)}(p; s, -s') \gamma_d \\ &= -\frac{1}{2} [\text{sgn}(s + s') + i \gamma_d \not{p}] e^{-|p| |s + s'|}. \end{aligned} \quad (5)$$

In the above equation ‘sgn’ stands for the ‘sign’ function, $|p|$ is the modulus of p_μ , and $\hat{p}_\mu \equiv \frac{p_\mu}{|p|}$. Notice that $s, s' > 0$ in \mathcal{M} , thus we may ignore the $\text{sgn}(s + s')$ in $\tilde{S}_F^{(1)}$, as that function is identically equal to 1 there.

Equipped with the expression for \tilde{S}_F , it is reassuring to check that the equations (1) are indeed satisfied. In the mixed Fourier representation, the first equation is tantamount to:

$$(\gamma_d \partial_s + i \not{p}) \tilde{S}_F(p; s, s') = \delta(s - s'). \quad (6)$$

Then it is immediate to see that

$$\begin{aligned} (\gamma_d \partial_s + i \not{p}) \tilde{S}_0(p; s, s') &= \delta(s - s') \\ (\gamma_d \partial_s + i \not{p}) \tilde{S}_1(p; s, s') &= -\delta(s + s') \gamma_d = 0, \end{aligned} \quad (7)$$

where in the last equation we used the fact that s and s' are in \mathcal{M} , thus $\delta(s + s') = 0$.

Regarding the bag boundary conditions, using the notation $\mathcal{P}_L \equiv \frac{1}{2}(1 - \gamma_d)$, we see that:

$$\begin{aligned} \lim_{s \rightarrow 0+} \mathcal{P}_L \tilde{S}_F(p; s, s') &= -\frac{1}{2} \lim_{s \rightarrow 0+} \mathcal{P}_L [\text{sgn}(s - s') e^{-|p| |s'|} + \text{sgn}(s + s') e^{-|p| |s'|}] \\ &= 0, \quad \forall s' > 0. \end{aligned} \quad (8)$$

Let us consider now the calculation of the one-loop vacuum polarization tensor, $\Pi_{\alpha\beta}$, in \mathcal{M} . Since we have the calculation response to an external constant electric field in mind as an application, a small momentum expansion is sufficient.

Adopting the convention that indices from the beginning of the Greek alphabet (α, β, \dots) will run from 0 to d , while the ones from the middle (μ, ν, \dots) will correspond to the range $0, 1, \dots, d-1$, we note that the mixed Fourier representation for $\Pi_{\alpha\beta}$, denoted $\tilde{\Pi}_{\alpha\beta}$, is:

$$\tilde{\Pi}_{\alpha\beta}(k; s, s') = -e^2 \int \frac{d^d p}{(2\pi)^d} \text{tr} \left[\tilde{S}_F(p; s', s) \gamma_\alpha \tilde{S}_F(p+k; s, s') \gamma_\beta \right], \quad (9)$$

and then we recall the MRE of S_F to obtain,

$$\tilde{\Pi}_{\alpha\beta}(k; s, s') = \sum_{l,m=0}^1 \tilde{\Pi}_{\alpha\beta}^{(lm)}(k; s, s'), \quad (10)$$

where,

$$\tilde{\Pi}_{\alpha\beta}^{(lm)}(k; s, s') = -e^2 \int \frac{d^d p}{(2\pi)^d} \text{tr} \left[\tilde{S}_F^{(l)}(p; s', s) \gamma_\alpha \tilde{S}_F^{(m)}(p+k; s, s') \gamma_\beta \right]. \quad (11)$$

The different terms in $\tilde{\Pi}_{\alpha\beta}$ are then obtained by evaluating the previous expression for the corresponding values of l and m .

To begin with, we note that $\tilde{\Pi}_{\alpha\beta}^{(00)}$ coincides with the vacuum polarization function in the absence of boundaries, or, what is equivalent, with boundaries at infinity. Then, to determine the contributions which truly come from the boundary, we should consider the remaining components $\tilde{\Pi}_{\alpha\beta}^{(lm)}$ with at least one of the indices l, m different from 0.

So, let us first consider the terms $\tilde{\Pi}^{(01)}$ and $\tilde{\Pi}^{(10)}$:

$$\begin{aligned} \tilde{\Pi}_{\alpha\beta}^{(01)}(k; s, s') &= \frac{e^2}{4} \int \frac{d^d p}{(2\pi)^d} \text{tr} \left[(\gamma_d \text{sgn}(s'-s) - i \not{p}) \gamma_\alpha (1 + i \gamma_d \gamma \cdot \widehat{(p+k)}) \gamma_\beta \right], \\ &\times \exp[-|p| |s-s'| - |p+k| |s+s'|] \end{aligned} \quad (12)$$

$$\begin{aligned} \tilde{\Pi}_{\alpha\beta}^{(10)}(k; s, s') &= \frac{e^2}{4} \int \frac{d^d p}{(2\pi)^d} \text{tr} \left[(1 + i \gamma_d \not{p}) \gamma_\alpha (\gamma_d \text{sgn}(s-s') - i \gamma \cdot \widehat{(p+k)}) \gamma_\beta \right] \\ &\times \exp[-|p| |s+s'| - |p+k| |s-s'|], \end{aligned} \quad (13)$$

where $(\widehat{p+k})_\mu \equiv (p+k)_\mu/|p+k|$.

It is immediate to notice that the results for $\tilde{\Pi}^{(01)}$ and $\tilde{\Pi}^{(10)}$ depend on the spacetime dimension $(d+1)$ being even or odd. Indeed, both contributions vanish when the spacetime dimension is even, since they involve the trace of an odd number (3 and 5) of γ matrices. Even for the case of an odd $(d+1)$, those traces can only be different from zero for $d=2$ and $d=4$. Namely, there is no parity breaking (to one-loop order) for any $d>4$ (even or odd).

In what follows, when considering these contributions, we shall assume that $d=2$, a physically interesting situation, since it corresponds to planar physics in a system with a border. This is potentially interesting when studying effective condensed matter physics models with Dirac fermions in $2+1$ dimensions coupled to a gauge field and confined to a half-plane.

Besides, notice that $\tilde{\Pi}^{(01)}$ and $\tilde{\Pi}^{(10)}$ yield parity-breaking contributions, since the trace of an odd number of matrices will necessarily introduce, in $2+1$ dimensions, a Levi-Civita tensor². This parity-breaking radiative effect is due to the explicit breaking of parity by the boundary condition [5].

Combining the two contributions, $\tilde{\Pi}^{(01)}$ and $\tilde{\Pi}^{(10)}$, into a single $\tilde{\Pi}^{odd} \equiv \tilde{\Pi}^{(01)} + \tilde{\Pi}^{(10)}$ tensor, we see (after taking the traces and performing a change of variables) that its μ, ν components are given by:

$$\begin{aligned} \tilde{\Pi}_{\mu\nu}^{odd}(k; s, s') &= i\frac{e^2}{2} \int \frac{d^2p}{(2\pi)^2} \left\{ [\varepsilon_{\mu\nu} \text{sgn}(s' - s) + \frac{\varepsilon_{\gamma\mu} p_\gamma k_\beta + \varepsilon_{\gamma\nu} p_\gamma k_\alpha}{|p| |p+k|} \right. \\ &\quad - \delta_{\mu\nu} \frac{p \wedge k}{|p| |p+k|}] \exp[-|p| |s-s'| - |p+k| |s+s'|] \\ &\quad + [\varepsilon_{\mu\nu} \text{sgn}(s' - s) - \frac{\varepsilon_{\gamma\mu} p_\gamma k_\nu + \varepsilon_{\gamma\nu} p_\gamma k_\mu}{|p| |p-k|} \\ &\quad \left. + \delta_{\mu\nu} \frac{p \wedge k}{|p| |p-k|}] \exp[-|p| |s-s'| - |p-k| |s+s'|] \right\} \quad (14) \end{aligned}$$

where we have introduced the notation $p \wedge q \equiv \varepsilon_{\mu\nu} p_\mu q_\nu$. This is still a rather complicated expression to deal with. However, for the response to a constant field it is enough to perform an expansion in powers of k , and evaluate the different contributions to $\tilde{\Pi}^{odd}$. The leading term when $k \rightarrow 0$ (required for the calculation of the induced current) is:

$$\tilde{\Pi}_{\mu\nu}^{odd}(0; s, s') = -i\frac{e^2}{2\pi} \frac{\text{sgn}(s' - s)}{(|s-s'| + |s+s'|)^2} \varepsilon_{\mu\nu} . \quad (15)$$

²That will not be so for the $\tilde{\Pi}^{(11)}$ term, which, as we shall see, is parity-conserving and different from zero for all d .

There are, of course, also components: $\tilde{\Pi}_{dd}^{odd}(k; s, s')$ and $\tilde{\Pi}_{d\mu}^{odd}(k; s, s')$. However, they vanish identically when $k \rightarrow 0$.

Let us now calculate the remaining term $\tilde{\Pi}_{\alpha\beta}^{(11)}$ in $d+1$ spacetime dimensions. Using (11) and (5) we see that it is given by:

$$\begin{aligned} \tilde{\Pi}_{\alpha\beta}^{(11)}(k; s, s') &= -\frac{e^2}{4} \int \frac{d^d p}{(2\pi)^d} \text{tr} \left[(1 - i\gamma_d \not{p}) \gamma_\alpha (1 + i\gamma_d (\widehat{p+k})) \gamma_\beta \right] \\ &\times \exp[-(|p| + |p+k|)|s+s'|], \end{aligned} \quad (16)$$

where $s, s' > 0$. To perform the trace operations in (16), we consider the cases $\tilde{\Pi}_{\mu\nu}^{(11)}$, $\tilde{\Pi}_{d\mu}^{(11)}$ and $\tilde{\Pi}_{dd}^{(11)}$ separately. We get,

$$\begin{aligned} \tilde{\Pi}_{\mu\nu}^{(11)}(k; s, s') &= -\frac{e^2}{4} r \int \frac{d^d p}{(2\pi)^d} \exp[-(|p| + |p+k|)|s+s'|] \\ &\times \left[\delta_{\mu\nu} - \hat{p}_\mu (\widehat{p+k})_\nu - \hat{p}_\nu (\widehat{p+k})_\mu + \hat{p} \cdot (\widehat{p+k}) \delta_{\mu\nu} \right], \end{aligned} \quad (17)$$

where r stands for the trace of the unit matrix in a $(d+1)$ -dimensional spacetime. In the $k \rightarrow 0$ limit,

$$\tilde{\Pi}_{\mu\nu}^{(11)}(0; s, s') = -\frac{e^2}{2} r \left(1 - \frac{1}{d} \right) \delta_{\mu\nu} I_0(s+s'), \quad (18)$$

where $I_0(s+s')$ may be obtained from the general integral,

$$I_n(t) = \int \frac{d^d p}{(2\pi)^d} \frac{\exp[-2|p||t|]}{|p|^n} = \frac{2^{1-2d+n} \Gamma(d-n)}{\pi^{d/2} \Gamma(d/2)} \frac{1}{|t|^{d-n}}. \quad (19)$$

For the $\tilde{\Pi}_{d\mu}^{(11)}$ component, we obtain from (16) the expression,

$$\tilde{\Pi}_{d\mu}^{(11)}(k; s, s') = -i\frac{e^2}{4} r \int \frac{d^d p}{(2\pi)^d} \exp[-(|p| + |p+k|)|s+s'|] \left[(\widehat{p+k})_\mu - \hat{p}_\mu \right], \quad (20)$$

which leads to $\tilde{\Pi}_{d\mu}^{(11)}(0; s, s') = 0$.

Analogously, for $\tilde{\Pi}_{dd}^{(11)}$:

$$\tilde{\Pi}_{dd}^{(11)}(k; s, s') = -\frac{e^2}{4} r \int \frac{d^d p}{(2\pi)^d} \exp[-(|p| + |p+k|)|s+s'|] \left[1 - \hat{p} \cdot (\widehat{p+k}) \right], \quad (21)$$

and also in this case we have $\tilde{\Pi}_{dd}^{(11)}(0; s, s') = 0$.

The small- k expressions for $\tilde{\Pi}_{dd}^{(11)}$ and $\tilde{\Pi}_{d\mu}^{(11)}$ are:

$$\tilde{\Pi}_{d\mu}^{(11)}(k; s, s') = -i \frac{e^2}{4} r \left(1 - \frac{1}{d}\right) k_\mu I_1(s + s'), \quad (22)$$

and

$$\tilde{\Pi}_{dd}^{(11)}(k; s, s') = -\frac{e^2}{4} r \left[-2|s + s'| \frac{1}{d} k^2 I_1(s + s') + \frac{1}{2} k^2 \left(1 - \frac{5}{d}\right) I_2(s + s') \right]. \quad (23)$$

where $I_1(s + s')$ and $I_2(s + s')$ are obtained from (19). One can also verify that for small k ,

$$\tilde{\Pi}_{dd}^{(00)}(k; s, s') = -\tilde{\Pi}_{dd}^{(11)}(k; s, -s') \quad (24)$$

and

$$\tilde{\Pi}_{d\mu}^{(00)}(k; s, s') = -\tilde{\Pi}_{d\mu}^{(11)}(k; s, -s') \text{sgn}(s' - s). \quad (25)$$

In the linear response approximation, the induced current can be written as follows:

$$j_\alpha(x_\perp, s) = -i \int dy_\perp ds' \Pi_{\alpha\beta}(x_\perp, y_\perp; s, s') A_\beta(y_\perp, s'), \quad (26)$$

where we follow the notation introduced in (2). We are interested in the particular case of a constant electric field with modulus E and normal to the wall. In the gauge $A_\alpha = 0$, $\alpha = 1, \dots, d$ we have,

$$A_0(y_\perp, s') = -E s' \quad (27)$$

and

$$j_\alpha(x_\perp, s) = i E \int dy_\perp^0 \int d^{d-1} y_\perp \int_0^\infty ds' \Pi_{\alpha 0}(x_\perp - y_\perp; s, s') s'. \quad (28)$$

Translation invariance in x_\perp implies of course that the current is independent of x_\perp , so $j_\alpha = j_\alpha(s)$ (as can be seen by a shift of variables).

Next we distinguish the cases of spacetime dimension $d + 1$ even or odd ($d = 2$). In the first case, since $\tilde{\Pi}^{odd} = \tilde{\Pi}^{(01)} + \tilde{\Pi}^{(10)} = 0$, and both $\tilde{\Pi}_{\mu\nu}^{(11)}$ and $\tilde{\Pi}_{\mu\nu}^{(00)}$ are proportional to $\delta_{\mu\nu}$, only the j_0 component is different from zero. On the other hand, when $d = 2$, both j_0 and j_1 may be non-vanishing, since

$\tilde{\Pi}_{l0}^{odd}(0; s, s')$ is different from 0. For j_0 (and any d), the contribution always comes only from $\tilde{\Pi}_{00}^{(11)}(0; s, s')$.

Now, recalling (15), we obtain for $d = 2$

$$j_1(s) = -E \frac{e^2}{2\pi} \int_0^\infty ds' \frac{\text{sgn}(s' - s)}{(|s - s'| + |s + s'|)^2}, \quad (29)$$

and evaluating the integral over s' :

$$j_1(s) = -\frac{e^2 E}{4\pi s}. \quad (30)$$

This result shows that there is indeed a non-trivial parity breaking effect due to the boundary: the induction of a current *transverse* to the electric field. That current is also concentrated on the wall ($s = 0$), where parity is maximally violated.

Analogously, from (18), and introducing an ultraviolet regularization through a small ϵ parameter, we obtain,

$$j_0(s) = -i \frac{e^2 E r(d-1)}{2d} \frac{2^{1-2d} \Gamma(d)}{\pi^{d/2} \Gamma(d/2)} \times \left\{ \int_0^\infty s' ds' \frac{1}{[(s - s')^2 + \epsilon^2]^d} + \int_0^\infty s' ds' \frac{1}{[(s + s')^2 + \epsilon^2]^d} \right\}, \quad (31)$$

for any d . When $d = 2$, IR divergences are present, which we deal with by means of an IR cutoff L . After some standard calculations, the leading contribution for a small ϵ is:

$$j_0(s) = -i \frac{e^2 E}{8\pi} \left[-\ln\left(\frac{s}{L}\right) + \frac{\pi s}{2\epsilon} \right]. \quad (32)$$

We see from (30) and (32) that the $j_1(s)$ component is divergent only on the boundary ($s = 0$). The $j_0(s)$ component, on the other hand, has a space-dependent UV divergence which vanishes on the wall.

Let mention that the Ward identity for $\Pi_{\alpha\beta}$, not evident in the different expressions we have obtained for $\Pi_{\alpha\beta}$, can be checked explicitly.

Finally, we mention that, had the boundary been located at an arbitrary point s^0 rather than at $s = 0$, the resulting $\tilde{\Pi}_{\alpha\beta}$ could have been easily

obtained from the one for $s = 0$ by the change of variables: $s \rightarrow s - s^0$ and $s' \rightarrow s' - s^0$ in $\tilde{\Pi}_{\alpha\beta}$. Namely:

$$\left[\tilde{\Pi}_{\alpha\beta}(k; s, s')\right]_{s_0 \neq 0} = \left[\tilde{\Pi}_{\alpha\beta}(k; s - s^0, s' - s^0)\right]_{s_0=0}. \quad (33)$$

The reason is that the fermion propagator in the new variables may be obtained by the same change of variables. That this is so can be checked, for example, by noting that both the differential equation and the bag condition 1 are then automatically satisfied on the new boundary.

It is then quite easy to obtain consider, for example, the expressions for the vacuum polarization function in the limit of a “wall at infinity’’, which corresponds to $s^0 \rightarrow -\infty$. One easily verifies that, for $s, s' > s^0$, all the terms $\tilde{\Pi}^{(lm)}$ with l or m different from 0 vanish. The result tends to the usual, no-reflection one.

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